Multigrid Convergent Principal Curvature Estimators in Digital Geometry

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Context

Differential quantities...

- for shape analysis, shape matching, ...
- for mathematical modeling of deformable objects (DIGITALSNOW project)

How to make an estimator?

- Experimental analysis of approximation errors on shapes with known Euclidean values
- Formal proof of convergence
- Computational cost & timing
Let us consider a family $X$ of smooth and compact subsets of $\mathbb{R}^d$. We denote shape $X$ as $X \in X$, and $D_h(X)$ the digitization of $X$ in a $d$-dimensional grid of resolution $h$. More precisely, we consider classical Gauss digitization defined as

$$D_h(X) \overset{def}{=} \left( \frac{1}{h} \cdot X \right) \cap \mathbb{Z}^d$$

where $\frac{1}{h} \cdot X$ is the uniform scaling of $X$ by factor $\frac{1}{h}$. Furthermore, the set $\partial X$ denotes the frontier of $X$ (i.e. its topological boundary). The $h$-boundary $\partial_h X$ is a $d - 1$-dimensional subset of $\mathbb{R}^d$, which is close to $\partial X$. 

Multigrid Convergence Framework
Multigrid convergence for local geometric quantities

Definition

A local discrete geometric estimator $\hat{E}$ of some geometric quantity $E$ is **multigrid convergent** for the family $X$ if and only if, for any $X \in X$, there exists a grid step $h_X > 0$ such that the estimate $\hat{E}(D_h(X), \hat{x}, h)$ is defined for all $\hat{x} \in \partial h X$ with $0 < h < h_X$, and for any $x \in \partial X$,

$$\forall \hat{x} \in \partial h X \text{ with } \|\hat{x} - x\|_{\infty} \leq h, |\hat{E}(D_h(X), \hat{x}, h) - E(X, x)| \leq \tau_{X,x}(h),$$

where $\tau_{X,x} : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ has null limit at 0. This function defines the **speed of convergence** of $\hat{E}$ toward $E$ at point $x$ of $X$. The convergence is **uniform** for $X$ when every $\tau_{X,x}$ is **bounded** from above by a function $\tau_X$ independent of $x \in \partial X$ with **null limit at 0**.
Family of curvature estimators

**Point clouds**
- Fitting a polynomial surface of degree at least 2 => convergence results but nothing with noised data
- Estimate orthogonal space with Voronoi diagram => convergence results but several parameters

**Triangulated meshes**
- Fitting
- Discrete method
- Integral invariant <= stability results when the kernel and the mesh sampling tend to zero

=> Most of them have no theoretical convergence guarantees even without noise.

**Digital data**
- Polynomial Fitting
- Binomial convolution
# 2D/3D Curvature Estimators

## 2D Experimentally convergent
- **MDCA estimator** [Roussillon, T. and Lachaud, J.O., 2011]
  - *Uses the most centered maximal Digital Circular Arc (DCA) to estimate the radius of the osculating circle.*

## 2D Theoretically & Experimentally convergent
- **BC curvature estimator** [Esbelin, H.A. and Malgouyres, R., 2009]
  - *Convergence speed in $O(h^{4/3})$*

## 3D Theoretically & Experimentally convergent
- **Curvature estimation using Polynomial fitting of osculating jets** [Cazals, F. and Pouget, M., 2005]
  - *Convergence speed in $O(\delta^2)$ with $\delta$ the density of points.*
Main contribution

Digital curvature estimators:
- defined in both 2D and 3D (mean and principal curvatures)
- easy to implement
- multigrid convergence is theoretically proved with an uniform convergence speed in $O(h^{\frac{1}{3}})$
- experimental validation of multigrid convergence
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Definition

Given $X \in \mathbb{R}^n$ and a radius $r \in \mathbb{R}^+$, the volumetric integral $V_r(x)$ at $x \in \partial X$ is given by

$$V_r(x) \overset{def}{=} \int_{B_r(x)} \chi(p) dp$$

where $B_r(x)$ is the Euclidean ball (kernel) with radius $r$ and center $x$ and $\chi(p)$ the characteristic function of $X$. In dimension 2, we simply denote $A_r(x)$ such quantity.

Local estimators $\tilde{\kappa}_r(x)$ and $\tilde{H}_r(x)$

$$\tilde{\kappa}_r(X, x) \overset{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}, \quad \tilde{H}_r(X, x) \overset{def}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$$

Then:

$$\tilde{\kappa}_r(X, x) = \kappa(X, x) + O(r), \quad \tilde{H}_r(X, x) = H(X, x) + O(r)$$
Proof process 2d & mean 3d curvature

\[ A_r(x) \rightarrow \text{Area}(D_h(B_r(x) \cap X), h) \]

Convergence of \( \hat{\kappa}_r(D_h(X), x, h) \) and \( \hat{H}_r(D_h(X'), x, h) \)

Convergence of \( \hat{\kappa}_r(D_h(X), \hat{x}, h) \) and \( \hat{H}_r(D_h(X'), \hat{x}, h) \)
Conclusion 2d & 3d mean curvature

Theorem (Uniform convergence of $\hat{H}_r$ along $\partial_h X$)

Let $X'$ be some convex shape of $\mathbb{R}^3$, with at least $C^2$-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0$ $\forall x \in \partial X'$, $\forall \hat{x} \in \partial_h X'$, $\|\hat{x} - x\|_\infty \leq h$

$$\forall 0 < h < r, \hat{H}_r(\partial_h X', \hat{x}, h) \overset{\text{def}}{=} \frac{8}{3r} - \frac{4\text{Vol}(B_{r/h}(\hat{x}) \cap \partial_h X', h)}{\pi r^4}.$$ 

Setting $r = k'h^{\frac{1}{3}}$, we have $|\hat{H}_r(\partial_h (X'), \hat{x}, h) - H(X', x)| \leq K'h^{\frac{1}{3}}$
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Principal curvatures with covariance matrix

**(p,q,s)-moments**

For non-negative integers p, q, and s, with a non-empty subset \( Y \) of \( \mathbb{R}^3 \)

\[
m_{p,q,s}(Y) \overset{\text{def}}{=} \iiint_Y x^p y^q z^s \, dxdydz
\]

**Covariance matrix**

For simplicity, \( A = B_R(x) \cap X \).

\[
J(A) = \begin{bmatrix}
m_{2,0,0}(A) & m_{1,1,0}(A) & m_{1,0,1}(A) \\
m_{1,1,0}(A) & m_{0,2,0}(A) & m_{0,1,1}(A) \\
m_{1,0,1}(A) & m_{0,1,1}(A) & m_{0,0,2}(A)
\end{bmatrix}
\]

\[
- \frac{1}{m_{0,0,0}(A)} \begin{bmatrix}
m_{1,0,0}(A) \\
m_{0,1,0}(A) \\
m_{0,0,1}(A)
\end{bmatrix} \otimes \begin{bmatrix}
m_{1,0,0}(A) \\
m_{0,1,0}(A) \\
m_{0,0,1}(A)
\end{bmatrix}^T
\]

\( \otimes \) denotes the usual tensor product in vector spaces.
**Lemma [Pottmann2007]**

Given a shape $X \in \mathbb{X}$, the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $J(A)$, where $A = B_R(x) \cap X$ and $x \in \partial X$, have the following Taylor expansion:

\[
\begin{align*}
\lambda_1 &= \frac{2\pi}{15} R^5 - \frac{\pi}{48} (3 \kappa^1(X, x) + \kappa^2(X, x)) R^6 + O(R^7) \\
\lambda_2 &= \frac{2\pi}{15} R^5 - \frac{\pi}{48} (\kappa^1(X, x) + 3 \kappa^2(X, x)) R^6 + O(R^7) \\
\lambda_3 &= \frac{19\pi}{480} R^5 - \frac{9\pi}{512} (\kappa^1(X, x) + \kappa^2(X, x)) R^6 + O(R^7)
\end{align*}
\]

where $\kappa^1(X, x)$ and $\kappa^2(X, x)$ denotes the principal curvatures of $\partial X$ at $x$.

**Local estimators $\tilde{\kappa}^1(X, x)$ and $\tilde{\kappa}^2(X, x)$**

\[
\begin{align*}
\tilde{\kappa}^1(X, x) &= \frac{6}{\pi R^6} (\tilde{\lambda}_2 - 3 \tilde{\lambda}_1) + \frac{8}{5 R} \\
\tilde{\kappa}^2(X, x) &= \frac{6}{\pi R^6} (\tilde{\lambda}_1 - 3 \tilde{\lambda}_2) + \frac{8}{5 R}
\end{align*}
\]
Proof process

Convergence of digital moments

\[ \hat{m}_{p,q,r}(D_h(B_R(x) \cap X), h) \]

Convergence of digital covariance matrix

\[ \hat{m}_{p,q,r}(D_h(B_R(x) \cap X), h) + [\text{Pottmann2007}] \rightarrow \hat{J}(D_h(B_R(x) \cap X), h) \]

Convergence of digital covariance matrix with position error.

Convergence of principal curvature estimators \( \hat{\kappa}_1 \) and \( \hat{\kappa}_2 \) along \( \partial_h X \)
Step 1a - Moment estimation

Digital \((p,q,s)\)-moments

We define the *digital* \((p, q, s)\)-moments \(\hat{m}_{p,q,s}(Z, h)\) of a subset \(Z\) of \(\mathbb{Z}^3\) at step \(h\) as:

\[
\hat{m}_{p,q,s}(Z, h) \overset{def}{=} h^{3+p+q+s} \sum_{(i,j,k) \in Z} i^p j^q k^s
\]

If \(Z = D_h(Y)\), \(Y\) a non-empty subset of \(\mathbb{R}^3\), and \(\sigma \overset{def}{=} p + q + s\) the order of the moment:

\[
\hat{m}_{p,q,s}(D_h(Y), h) = m_{p,q,s}(Y) + O(h^{\mu\sigma}).
\]

- \(\mu\sigma = 1\) in general convex case
- \(\mu\sigma = \frac{38}{25} - \epsilon\) [Krätzel1991]
- \(\mu\sigma = \frac{66}{43} - \epsilon\) [Müller1999]
Step 1b - Moment estimation

\[ \| \hat{m}_{p,q,s}(D_h(B_R(x) \cap X), h) - m_{p,q,s}(B_R(x) \cap X) \| = K'(r) h^{\mu \sigma} \]

\[ \hat{m}_{p,q,s}(D_h(B_R(x) \cap X), h) = R^{3+\sigma} \hat{m}_{p,q,s} \left( D_{h/R}(B_1 \left( \frac{1}{R} \cdot x \right) \cap \frac{1}{R} \cdot X), \frac{h}{R} \right) \]

\[ \| \hat{m}_{p,q,s}(D_h(B_R(x) \cap X), h) - m_{p,q,s}(B_R(x) \cap X) \| = KR^{3+\sigma-\mu \sigma} h^{\mu \sigma} \]

with \( \mu_{\sigma} \geq 1 \).

Proof hints

- Rescale shapes \( Z \) to only a unit ball \( B_1 \)


Step 2 - Covariance matrix estimation

\[ \hat{m}_{p,q,r}(D_h(B_R(x) \cap X), h) + [Pottmann2007] \]

Reminder:

\[ \hat{m}_{p,q,s}(Z, h) \overset{def}{=} h^{3+p+q+s} M_{p,q,s}(Z) \]

We can define:

Digital covariance matrix estimator \( \hat{J}(Z, h) \) of a digital shape \( Z \) at point \( x \in \mathbb{R}^3 \) and step \( h \):

\[
\hat{J}(Z, h) \overset{def}{=} \begin{bmatrix}
\hat{m}_{2,0,0}(Z, h) & \hat{m}_{1,1,0}(Z, h) & \hat{m}_{1,0,1}(Z, h) \\
\hat{m}_{1,1,0}(Z, h) & \hat{m}_{0,2,0}(Z, h) & \hat{m}_{0,1,1}(Z, h) \\
\hat{m}_{1,0,1}(Z, h) & \hat{m}_{0,1,1}(Z, h) & \hat{m}_{0,0,2}(Z, h)
\end{bmatrix}
- \frac{1}{\hat{m}_{0,0,0}(Z, h)} \begin{bmatrix}
\hat{m}_{1,0,0}(Z, h) \\
\hat{m}_{0,1,0}(Z, h) \\
\hat{m}_{0,0,1}(Z, h)
\end{bmatrix} \otimes \begin{bmatrix}
\hat{m}_{1,0,0}(Z, h) \\
\hat{m}_{0,1,0}(Z, h) \\
\hat{m}_{0,0,1}(Z, h)
\end{bmatrix}^T
\]

Theorem (Multigrid convergence of digital covariance matrix)

Let \( X \in \mathbb{X} \). Then, there exists some constant \( h_X \), such that for any grid step \( 0 < h < h_X \), for arbitrary \( x \in \mathbb{R}^3 \), for arbitrary \( R \geq h \), we have:

\[
\| \hat{J}(D_h(B_R(x) \cap X), h) - J(B_R(x) \cap X) \| \leq O(R^{5-\mu_0} h^{\mu_0}) + O(R^{5-\mu_1} h^{\mu_1}) + O(R^{5-\mu_2} h^{\mu_2})
\]

The constants hidden in the big O do not depend on the shape size or geometry.
Step 3a - Positioning error on moments

For any subset \( X \subset \mathbb{R}^3 \) and for any vector \( t \) with norm \( t \defeq \|t\|_2 \leq R \), we have for \( 0 \leq p + q + s \leq 2 \):

\[
m_{p,q,s}(B_R(x + t) \cap X) = m_{p,q,s}(B_R(x) \cap X) + \sum_{i=0}^{p+q+s} O(\|x\|^i t R^{2+p+q+s-i}).
\]

**Proof hints**

- For zeroth order moment, we use the volume of the blue part
- For first order moment, we use the fact that the centered 1,0,0-moment is maximized by the x-positive half-ball of the blue part
- For second order moment, we use the fact that the centered 2,0,0-moment is maximized by the ball
Step 3b - Positioning error on covariance matrix

Theorem (Multigrid convergence of digital covariance matrix with position error)

Let $X \in \mathbb{X}$. Then, there exists some constant $h_X$, such that for any grid step $0 < h < h_X$, for arbitrary $R \geq h$, for any $x \in \partial X$ and any $\hat{x} \in \partial D_h(X)$, $\|x - \hat{x}\|_{\infty} \leq h$, we have:

$$\| \hat{J}(D_h(A(R, \hat{x})), h) - J(A(R, x)) \| \leq \| \hat{J}(D_h(A(R, \hat{x})), h) - J(A(R, \hat{x}))\|$$
$$+ \| J(A(R, \hat{x})) - J(A(R, x)) \|$$
$$\leq \sum_{i=0}^{2} O(R^{5-\mu_i} h^{\mu_i}) + O(\|x - \hat{x}\| R^4)$$

The constants hidden in the big O do not depend on the shape size or geometry.
Step 3c - Convergence of digital principal curvature estimators

Theorem (Lidskii-Weyl inequality) [Stewart1990][Bhatia1997]

If \( \lambda_i(B) \) denotes the ordered eigenvalues of some symmetric matrix \( B \) and \( \lambda_i(B + E) \) the ordered eigenvalues of some symmetric matrix \( B + E \), then \( \max_i |\lambda_i(B) - \lambda_i(B + E)| \leq \|E\| \).

Let \( Z \) be a digital shape, \( x \) some point of \( \mathbb{R}^3 \) and \( h > 0 \) a gridstep. For \( R \geq h \), we define the integral principal curvature estimators \( \hat{\kappa}^1_R \) and \( \hat{\kappa}^2_R \) of \( Z \) at point \( y \in \mathbb{R}^3 \) and step \( h \) as

\[
\hat{\kappa}^1_R(Z, y, h) = \frac{6}{\pi R^6} (\hat{\lambda}_2 - 3\hat{\lambda}_1) + \frac{8}{5R},
\]

\[
\hat{\kappa}^2_R(Z, y, h) = \frac{6}{\pi R^6} (\hat{\lambda}_1 - 3\hat{\lambda}_2) + \frac{8}{5R},
\]

where \( \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \) are the two greatest eigenvalues of \( \hat{J}(B_R/h(\frac{1}{h} \cdot y) \cap Z, h) \)
Step 3c - Convergence of digital principal curvature estimators

Theorem (Uniform convergence of principal curvature estimators $\hat{\kappa}_R^1$ and $\hat{\kappa}_R^2$ along $\partial_h X$)

Let $X \in \mathbb{X}$. For $i \in \{1, 2\}$, recall that $\kappa^i(X, x)$ is the $i$-th principal curvature of $X$ at boundary point $x$. Then, $\exists h_X \in \mathbb{R}^+$, for any $h \leq h_X$, we have

$$\forall x \in \partial X, \forall \hat{x} \in \partial_h X, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$

$$|\hat{\kappa}_R^i(D_h(X), \hat{x}, h) - \kappa^i(X, x)| \leq O(R) + O(h/R^2) + \sum_{i=0}^{2} O(h^{\mu_i}/R^{1+\mu_i}).$$

$$\mu = 1, \quad \frac{R}{kh} = k h^\alpha$$

$\Rightarrow \alpha_m = \frac{1}{3}$ $\Rightarrow |\hat{\kappa}_R^i(D_h(X), \hat{x}, h) - \kappa^i(X, x)| \leq Kh^{\frac{1}{3}}$
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Experimentation

Experimental Settings

- Family of Euclidean shapes (implicit, parametric) with exact curvature information
- Digitization process at resolution $h$
- Error metrics
  - Worst-case $l_\infty$ error: maximum of absolute difference value
    $\max_{\hat{x} \in \partial_h X, x \in \partial X} (|\hat{\kappa}_r(D_h(X), \hat{x}, h) - \kappa(X, x)|)$
  - Quadratic $l_2$ error
Validation of $\alpha$ parameter

Convolution kernel radius

$r = k h^{\alpha}$
Validation of $\alpha$ parameter

Convolution kernel radius

$r = k h^\alpha$
Comparison

![Graph showing comparison of BC, MDCA, MDSS, and II estimators.](image)

- **BC**: Red line
- **MDCA**: Green line
- **MDSS**: Pink line
- **II**: Blue line

The graphs illustrate the $L_\infty$ error as a function of noise level and step size $h$. The $O(h^{1/3})$ line indicates the expected convergence rate of the estimators. The graphs demonstrate the robustness and accuracy of the estimators across different noise levels and step sizes.
Comparison

Jet Fitting

$O(h^{1/3})$

Jet Fitting

$O(h^{1/3})$

Jet Fitting

$O(h^{1/3})$

Jet Fitting

$O(h^{1/3})$
Mean curvature and principal directions
Optimizations with convolution

- Open-source C++ library
- Geometry structures, algorithm & tools for digital data
- [http://libdgtal.org](http://libdgtal.org)

Optimization with displacement masks

Complexity:
- Without optimization: $O((r/h)^d)$
- With optimization: $O((r/h)^{d-1})$
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Conclusion & Future work

- Integral Invariant is perfect for digital geometry
- A unique estimator for both 2D and 3D
- Easy to implement
- Fast computation with masks
- Convergent with a least a uniform convergence speed in $O(h^{\frac{1}{3}})$
- Needs a parameter ($r$ for the kernel radius)

Future work

- Scale-Space analysis
- Feature detection
Choice of radius