Multigrid Convergent Principal Curvature Estimators in Digital Geometry

David Coeurjolly[1] Jacques-Olivier Lachaud[2,3] Jérémy Levallois[1,2]

[1] Université de Lyon, CNRS
INSA-Lyon, LIRIS, UMR5205, F-69621, France

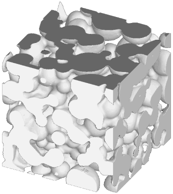
[2] Université de Savoie, CNRS
LAMA, UMR5127, F-73776, France

[2] Université Grenoble-Alpes, CNRS
LJK, UMR5224, F-38041, France

Table of content

Plan

Context

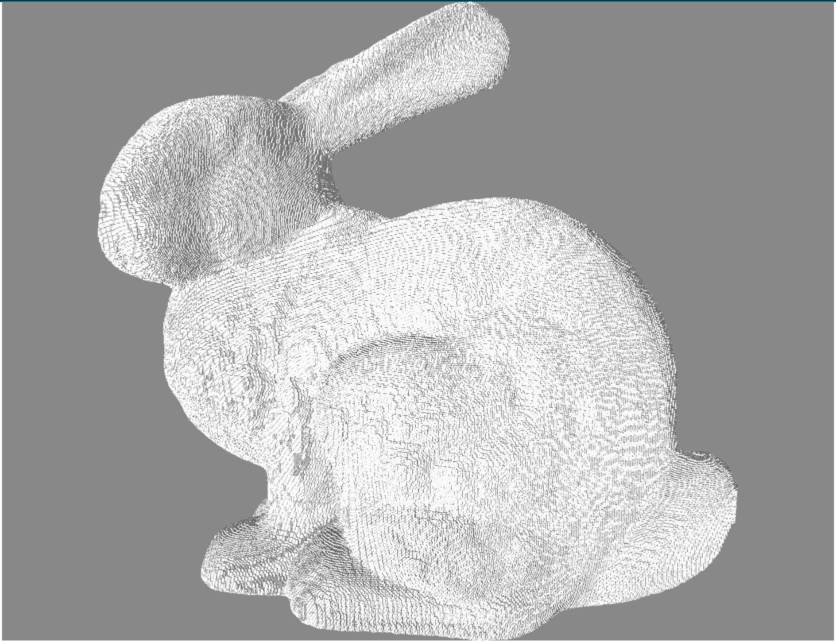


Differential quantities...

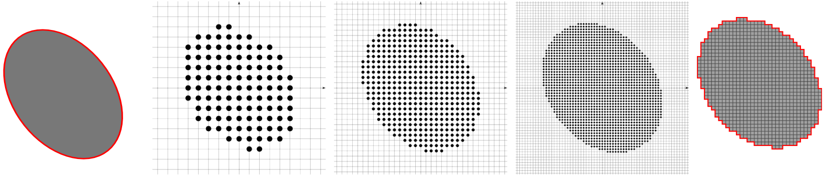
- for shape analysis, shape matching, ...
- for mathematical modeling of deformable objects (DIGITALSNOW project)

How to make an estimator ?

- Experimental analysis of approximation errors on shapes with known Euclidean values
- Formal proof of convergence
- Computational cost & timing



Multigrid Convergence Framework



Let us consider a **family** $\mathbb{X}$ of smooth and compact subsets of $\mathbb{R}^d$. We denote **shape** X as $X \in \mathbb{X}$, and $\mathtt{D}_h(X)$ the **digitization** of X in a $d-$dimensional grid of resolution h. More precisely, we consider classical Gauss digitization defined as

$$\mathtt{D}_h(X) \stackrel{def}{=} \left(\frac{1}{h} \cdot X\right) \cap \mathbb{Z}^d$$

where $\frac{1}{h} \cdot X$ is the uniform scaling of X by factor $\frac{1}{h}$. Furthermore, the set ∂X denotes the **frontier** of X (i.e. its topological boundary). The h-**boundary** $\partial_h X$ is a $d-1$-dimensional subset of $\mathbb{R}^d$, which is close to ∂X.

Multigrid convergence for local geometric quantities

Definition

A local discrete geometric estimator $\hat{E}$ of some geometric quantity E is ***multigrid convergent*** for the family $\mathbb{X}$ if and only if, for any $X \in \mathbb{X}$, there exists a grid step $h_X > 0$ such that the estimate $\hat{E}(\mathtt{D}_h(X), \hat{x}, h)$ is defined for all $\hat{x} \in \partial_h X$ with $0 < h < h_X$, and for any $x \in \partial X$,

$$\forall \hat{x} \in \partial_h X \text{ with } \|\hat{x} - x\|_\infty \leq h, |\hat{E}(\mathtt{D}_h(X), \hat{x}, h) - E(X, x)| \leq \tau_{X,x}(h),$$

where $\tau_{X,x} : \mathbb{R}^+ \setminus \{0\} \to \mathbb{R}^+$ has null limit at 0. This function defines the **speed of convergence** of $\hat{E}$ toward E at point x of X. The convergence is ***uniform*** for X when every $\tau_{X,x}$ is **bounded** from above by a function τ_X independent of $x \in \partial X$ with **null limit at 0**.

Family of curvature estimators

Point clouds

- Fitting a polynomial surface of degree at least 2 => convergence results but nothing with noised data
- Estimate orthogonal space with Voronoi diagram => convergence results but several parameters

Triangulated meshes

- Fitting
- Discrete method
- Integral invariant <= stability results when the kernel and the mesh sampling tend to zero

=> Most of them have no theoritical convergence guarantees even without noise.

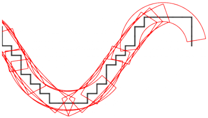
Digital data

- Polynomial Fitting
- Binomial convolution

2D/3D Curvature Estimators

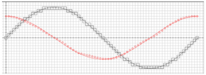
2D Experimentally convergent

- MDCA estimator [Roussillon, T. and Lachaud, J.O., 2011]
  Uses the most centered maximal Digital Circular Arc (DCA) to estimate the radius of the osculating circle.



2D Theoretically & Experimentally convergent

- BC curvature estimator [Esbelin, H.A. and Malgouyres, R., 2009]
  convergence speed in $O(h^{\frac{4}{9}})$



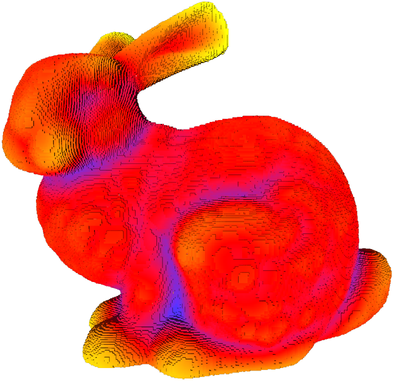
3D Theoretically & Experimentally convergent

- Curvature estimation using Polynomial fitting of osculating jets [Cazals, F. and Pouget, M., 2005]
  convergence speed in $O(\delta^2)$ *with* δ *the density of points.*

Main contribution

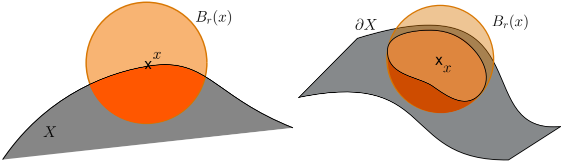
Digital curvature estimators :

- defined in both 2D and 3D (mean and principal curvatures)
- easy to implement
- multigrid convergence is theoretically proved with an uniform convergence speed in $O(h^{\frac{1}{3}})$
- experimental validation of multigrid convergence



Plan

Integration based surface feature



Definition

Given $X \in \mathbb{X}$ and a radius $r \in \mathbb{R}^{+*}$, the volumetric integral $V_r(x)$ at $x \in \partial X$ is given by

$$V_r(x) \stackrel{def}{=} \int_{B_r(x)} \chi(p)dp$$

where $B_r(x)$ is the Euclidean ball (kernel) with radius r and center x and $\chi(p)$ the characteristic function of X. In dimension 2, we simply denote $A_r(x)$ such quantity.

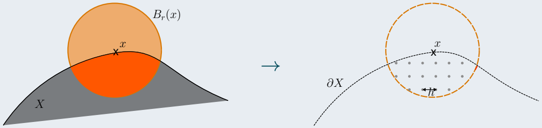
Local estimators $\tilde{\kappa}_r(x)$ and $\tilde{H}_r(x)$

$$\tilde{\kappa}_r(X,x) \stackrel{def}{=} \frac{3\pi}{2r} - \frac{3A_r(x)}{r^3}, \quad \tilde{H}_r(X,x) \stackrel{def}{=} \frac{8}{3r} - \frac{4V_r(x)}{\pi r^4}$$

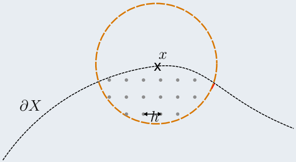
Then :

$$\tilde{\kappa}_r(X,x) = \kappa(X,x) + O(r), \quad \tilde{H}_r(X,x) = H(X,x) + O(r)$$

Proof process 2d & mean 3d curvature

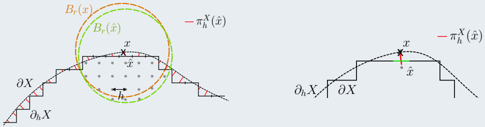


$A_r(x) \rightarrow \widehat{\mathrm{Area}}(\mathtt{D}_h(B_r(x) \cap X), h)$



$+$ [Pottmann2009]

Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), x, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), x, h)$



Convergence of $\hat{\kappa}_r(\mathtt{D}_h(X), \hat{x}, h)$ and $\hat{H}_r(\mathtt{D}_h(X'), \hat{x}, h)$

Conclusion 2d & 3d mean curvature

Theorem (Uniform convergence of $\hat{H}_r$ along $\partial_h X$)

Let X' be some convex shape of $\mathbb{R}^3$, with at least C^2-boundary and bounded curvature. Then, $\exists h_0 \in \mathbb{R}^+$, for any $h \leq h_0$ $\forall x \in \partial X', \forall \hat{x} \in \partial_h X'$, $\|\hat{x} - x\|_\infty \leq h$

$$\forall 0 < h < r, \hat{H}_r(\partial_h X', \hat{x}, h) \stackrel{def}{=} \frac{8}{3r} - \frac{4\widehat{\mathrm{Vol}}(B_{r/h}(\hat{x}) \cap \partial_h X', h)}{\pi r^4}.$$

Setting $r = k' h^{\frac{1}{3}}$, we have $|\hat{H}_r(\mathtt{D}_h(X'), \hat{x}, h) - H(X', x)| \leq K' h^{\frac{1}{3}}$

Plan

Principal curvatures with covariance matrix

(p,q,s)-moments

For non negative integers p, q and s, with a non-empty subset Y of $\mathbb{R}^3$

$$m_{p,q,s}(Y) \stackrel{def}{=} \iiint_Y x^p y^q z^s dxdydz$$

Covariance matrix

For simplicity, $A = B_R(x) \cap X$.

$$J(A) = \begin{bmatrix} m_{2,0,0}(A) & m_{1,1,0}(A) & m_{1,0,1}(A) \\ m_{1,1,0}(A) & m_{0,2,0}(A) & m_{0,1,1}(A) \\ m_{1,0,1}(A) & m_{0,1,1}(A) & m_{0,0,2}(A) \end{bmatrix} - \frac{1}{m_{0,0,0}(A)} \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix} \otimes \begin{bmatrix} m_{1,0,0}(A) \\ m_{0,1,0}(A) \\ m_{0,0,1}(A) \end{bmatrix}^T$$

$\otimes$ denotes the usual tensor product in vector spaces.

Principal curvatures with covariance matrix

Lemma [Pottmann2007]

Given a shape $X \in \mathbb{X}$, the eigenvalues λ_1, λ_2, λ_3 of $J(A)$, where $A = B_R(x) \cap X$ and $x \in \partial X$, have the following Taylor expansion :

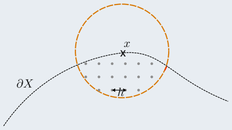
$$\begin{aligned}
\lambda_1 &= \frac{2\pi}{15} R^5 - \frac{\pi}{48}(3\kappa^1(X,x) + \kappa^2(X,x))R^6 + O(R^7) \\
\lambda_2 &= \frac{2\pi}{15} R^5 - \frac{\pi}{48}(\kappa^1(X,x) + 3\kappa^2(X,x))R^6 + O(R^7) \\
\lambda_3 &= \frac{19\pi}{480} R^5 - \frac{9\pi}{512}(\kappa^1(X,x) + \kappa^2(X,x))R^6 + O(R^7)
\end{aligned}$$

where $\kappa^1(X,x)$ and $\kappa^2(X,x)$ denotes the principal curvatures of ∂X at x.

Local estimators $\tilde{\kappa}^1(X,x)$ and $\tilde{\kappa}^2(X,x)$

$$\begin{aligned}
\tilde{\kappa}^1(X,x) &= \frac{6}{\pi R^6}(\tilde{\lambda}_2 - 3\tilde{\lambda}_1) + \frac{8}{5R} \\
\tilde{\kappa}^2(X,x) &= \frac{6}{\pi R^6}(\tilde{\lambda}_1 - 3\tilde{\lambda}_2) + \frac{8}{5R}
\end{aligned}$$

Proof process

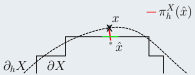


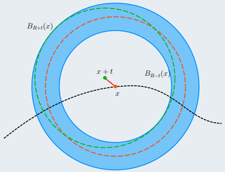
$\rightarrow \quad \hat{m}_{p,q,r}(\mathtt{D}_h(B_R(x) \cap X), h)$

Convergence of digital moments

$\hat{m}_{p,q,r}(\mathtt{D}_h(B_R(x) \cap X), h)$ + [Pottmann2007] $\rightarrow \quad \hat{J}(\mathtt{D}_h(B_R(x) \cap X), h)$

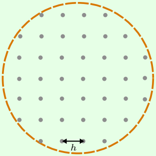
Convergence of digital covariance matrix





Convergence of digital covariance matrix with position error.
Convergence of principal curvature estimators $\hat{\kappa}^1_R$ and $\hat{\kappa}^2_R$ along $\partial_h X$

Step 1a - Moment estimation



Digital (p,q,s)-moments

We define the *digital* (p, q, s)*-moments* $\hat{m}_{p,q,s}(Z, h)$ of a subset Z of $\mathbb{Z}^3$ at step h as :

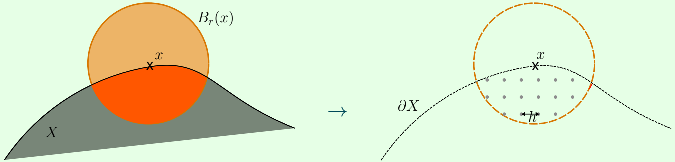
$$\hat{m}_{p,q,s}(Z,h) \stackrel{def}{=} h^{3+p+q+s} \sum_{(i,j,k)\in Z} i^p j^q k^s$$

If $Z = \mathtt{D}_h(Y)$, Y a non-empty subset of $\mathbb{R}^3$, and $\sigma \stackrel{def}{=} p+q+s$ the order of the moment :

$$\hat{m}_{p,q,s}(\mathtt{D}_h(Y),h) = m_{p,q,s}(Y) + O(h^{\mu_\sigma}).$$

- $\mu_\sigma = 1$ in general convex case
- $\mu_\sigma = \frac{38}{25} - \epsilon$ [Krätzel1991]
- $\mu_\sigma = \frac{66}{43} - \epsilon$ [Müller1999]

Step 1b - Moment estimation



$$\|\hat{m}_{p,q,s}(\mathtt{D}_h(B_R(x) \cap X), h) - m_{p,q,s}(B_R(x) \cap X)\| = K'(r)h^{\mu_\sigma}$$

$$\hat{m}_{p,q,s}(\mathtt{D}_h(B_R(x) \cap X), h) = R^{3+\sigma}\hat{m}_{p,q,s}\left(\mathtt{D}_{h/R}(B_1(\frac{1}{R} \cdot x) \cap \frac{1}{R} \cdot X), \frac{h}{R}\right)$$

$$\ldots$$

$$\|\hat{m}_{p,q,s}(\mathtt{D}_h(B_R(x) \cap X), h) - m_{p,q,s}(B_R(x) \cap X)\| = KR^{3+\sigma-\mu_\sigma}h^{\mu_\sigma}$$

with $\mu_\sigma \geq 1$.

Proof hints

- Rescale shapes Z to only a unit ball B_1

Step 2 - Covariance matrix estimation

$\hat{m}_{p,q,r}(\mathtt{D}_h(B_R(x) \cap X), h)$ + [Pottmann2007]

Convergence of $\hat{J}(\mathtt{D}_h(Z), x, h)$

Reminder :

$$\hat{m}_{p,q,s}(Z,h) \stackrel{def}{=} h^{3+p+q+s} M_{p,q,s}(Z)$$

We can define :
Digital covariance matrix estimator $\hat{J}(Z,h)$ of a digital shape Z at point $x \in \mathbb{R}^3$ and step h :

$$\hat{J}(Z,h) \stackrel{def}{=} \begin{bmatrix} \hat{m}_{2,0,0}(Z,h) & \hat{m}_{1,1,0}(Z,h) & \hat{m}_{1,0,1}(Z,h) \\ \hat{m}_{1,1,0}(Z,h) & \hat{m}_{0,2,0}(Z,h) & \hat{m}_{0,1,1}(Z,h) \\ \hat{m}_{1,0,1}(Z,h) & \hat{m}_{0,1,1}(Z,h) & \hat{m}_{0,0,2}(Z,h) \end{bmatrix} - \frac{1}{\hat{m}_{0,0,0}(Z,h)} \begin{bmatrix} \hat{m}_{1,0,0}(Z,h) \\ \hat{m}_{0,1,0}(Z,h) \\ \hat{m}_{0,0,1}(Z,h) \end{bmatrix} \otimes \begin{bmatrix} \hat{m}_{1,0,0}(Z,h) \\ \hat{m}_{0,1,0}(Z,h) \\ \hat{m}_{0,0,1}(Z,h) \end{bmatrix}^T$$

Theorem (Multigrid convergence of digital covariance matrix)

Let $X \in \mathbb{X}$. Then, there exists some constant h_X, such that for any grid step $0 < h < h_X$, for arbitrary $x \in \mathbb{R}^3$, for arbitrary $R \geq h$, we have :

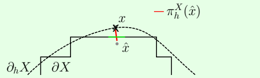
$$\|\hat{J}(\mathtt{D}_h(B_R(x) \cap X), h) - J(B_R(x) \cap X)\| \leq O(R^{5-\mu_0} h^{\mu_0}) + O(R^{5-\mu_1} h^{\mu_1}) + O(R^{5-\mu_2} h^{\mu_2})$$

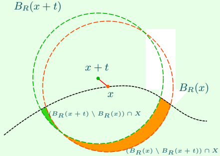
The constants hidden in the big O do not depend on the shape size or geometry.

Step 3a - Positioning error on moments

For any subset $X \subset \mathbb{R}^3$ and for any vector $\mathbf{t}$ with norm $t \stackrel{def}{=} \|\mathbf{t}\|_2 \leq R$, we have for $0 \leq p+q+s \leq 2$:

$$m_{p,q,s}(B_R(x+\mathbf{t}) \cap X) = m_{p,q,s}(B_R(x) \cap X) + \sum_{i=0}^{p+q+s} O(\|x\|^i t R^{2+p+q+s-i}).$$

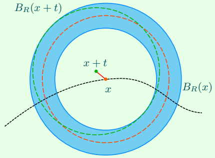




(a)

Proof hints

- For zeroth order moment, we use the volume of the blue part
- For first order moment, we use the fact that the centered 1,0,0-moment is maximized by the x-positive half-ball of the blue part
- For second order moment, we use the fact that the centered 2,0,0-moment is maximized by the ball



(b)

Step 3b - Positioning error on covariance matrix

Theorem (Multigrid convergence of digital covariance matrix with position error)

Let $X \in \mathbb{X}$. Then, there exists some constant h_X, such that for any grid step $0 < h < h_X$, for arbitrary $R \geq h$, for any $x \in \partial X$ and any $\hat{x} \in \partial \mathtt{D}_h(X)$, $\|x - \hat{x}\|_\infty \leq h$, we have :

$$
\begin{aligned}
\|\hat{J}(\mathtt{D}_h(A(R,\hat{x})),h) - J(A(R,x))\| &\leq \|\hat{J}(\mathtt{D}_h(A(R,\hat{x})),h) - J(A(R,\hat{x}))\| \\
&\quad + \|J(A(R,\hat{x})) - J(A(R,x))\| \\
&\leq \sum_{i=0}^{2} O(R^{5-\mu_i} h^{\mu_i}) + O(\|x - \hat{x}\| R^4)
\end{aligned}
$$

The constants hidden in the big O do not depend on the shape size or geometry.

Step 3c - Convergence of digital principal curvature estimators

Theorem (Lidskii-Weyl inequality) [Stewart1990][Bhatia1997]

If $\lambda_i(B)$ denotes the ordered eigenvalues of some symmetric matrix B and $\lambda_i(B+E)$ the ordered eigenvalues of some symmetric matrix $B+E$, then $\max_i |\lambda_i(B) - \lambda_i(B+E)| \leq \|E\|$.

Let Z be a digital shape, x some point of $\mathbb{R}^3$ and $h > 0$ a gridstep. For $R \geq h$, we define the *integral principal curvature estimators* $\hat{\kappa}^1_R$ *and* $\hat{\kappa}^2_R$ of Z at point $y \in \mathbb{R}^3$ and step h as

$$\begin{aligned} \hat{\kappa}^1_R(Z,y,h) &= \frac{6}{\pi R^6}(\hat{\lambda}_2 - 3\hat{\lambda}_1) + \frac{8}{5R}, \\ \hat{\kappa}^2_R(Z,y,h) &= \frac{6}{\pi R^6}(\hat{\lambda}_1 - 3\hat{\lambda}_2) + \frac{8}{5R}, \end{aligned}$$

where $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are the two greatest eigenvalues of $\hat{J}(B_{R/h}(\frac{1}{h} \cdot y) \cap Z, h))$

Step 3c - Convergence of digital principal curvature estimators

Theorem (Uniform convergence of principal curvature estimators $\hat{\kappa}_R^1$ and $\hat{\kappa}_R^2$ along $\partial_h X$)

Let $X \in \mathbb{X}$. For $i \in \{1, 2\}$, recall that $\kappa^i(X, x)$ is the i-th principal curvature of X at boundary point x. Then, $\exists h_X \in \mathbb{R}^+$, for any $h \leq h_X$, we have

$$\forall x \in \partial X, \forall \hat{x} \in \partial_h X, \|\hat{x} - x\|_\infty \leq h \Rightarrow$$
$$|\hat{\kappa}_R^i(\mathtt{D}_h(X), \hat{x}, h) - \kappa^i(X, x)| \leq \quad O(R) + O(h/R^2) + \textstyle\sum_{i=0}^{2} O(h^{\mu_i}/R^{1+\mu_i}).$$

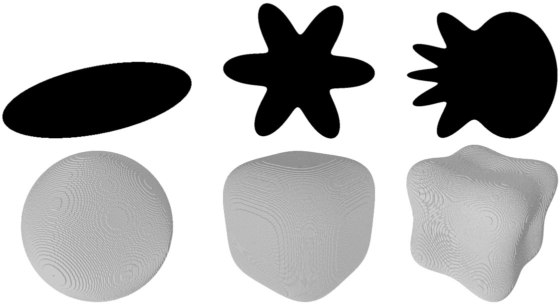
$$\left.\begin{array}{l}\mu = 1\\ R = kh^\alpha\end{array}\right\} \Rightarrow \alpha_m = \frac{1}{3} \Rightarrow |\hat{\kappa}_R^i(\mathtt{D}_h(X), \hat{x}, h) - \kappa^i(X, x)| \leq Kh^{\frac{1}{3}}$$

Plan

Experimentation

Experimental Settings

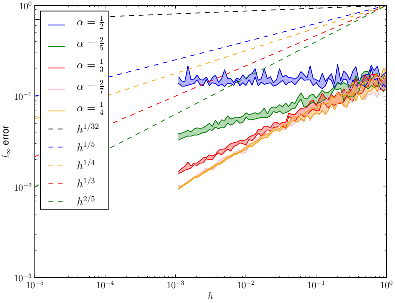
- Family of Euclidean shapes (implicit, parametric) with *exact* curvature information
- Digitization process at resolution h
- Error metrics
  - Worst-case l_∞ error : maximum of absolute difference value $\max_{\hat{x}\in\partial_h X, x\in\partial X}(|\hat{\kappa}_r(\mathtt{D}_h(X),\hat{x},h)-\kappa(X,x)|)$
  - Quadratic l_2 error

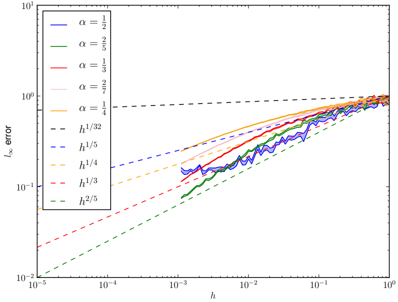


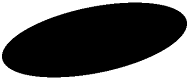
Validation of α parameter

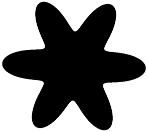
Convolution kernel radius

$r = kh^{\alpha}$





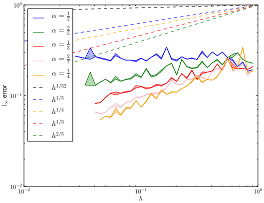


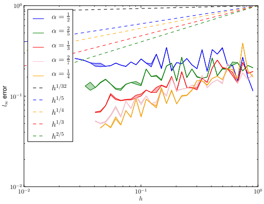


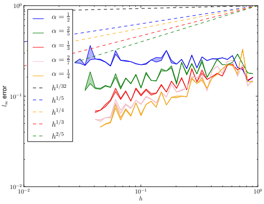
Validation of α parameter

Convolution kernel radius

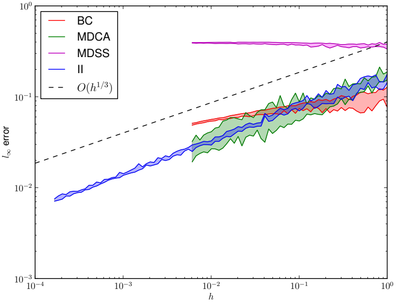
$r = kh^{\alpha}$

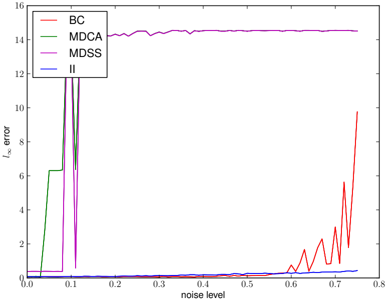


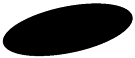


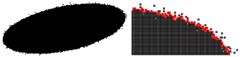


Comparison

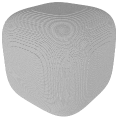

BC
MDCA
MDSS
II
$O(h^{1/3})$
l_∞ error
h

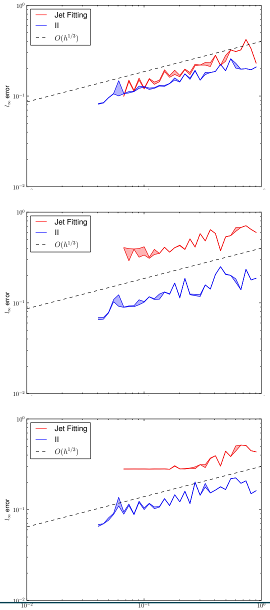

BC
MDCA
MDSS
II
l_∞ error
noise level

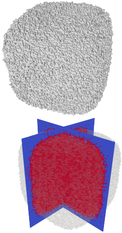


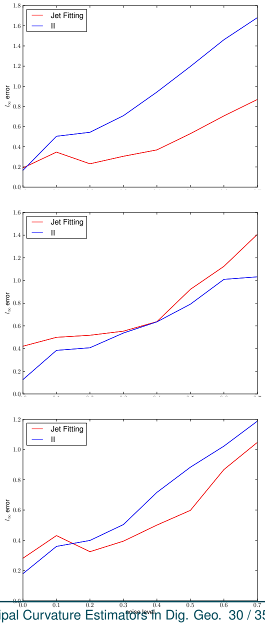


Comparison

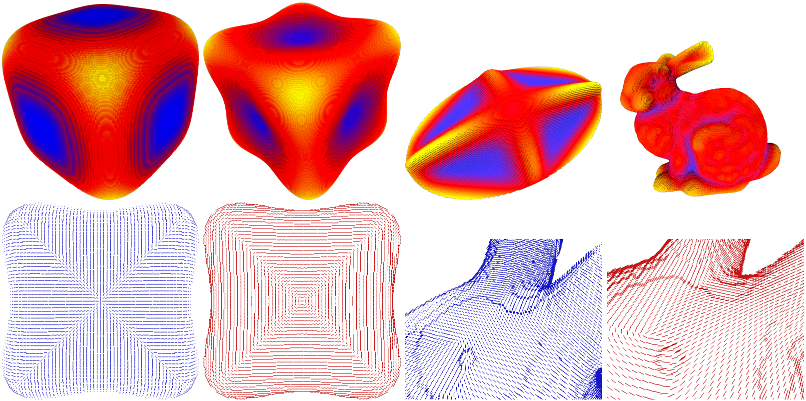



Jet Fitting
II
$O(h^{1/3})$
l_∞ error
Jet Fitting
II
$O(h^{1/3})$
l_∞ error
Jet Fitting
II
$O(h^{1/3})$
l_∞ error
h




Jet Fitting
II
l_∞ error
Jet Fitting
II
l_∞ error
Jet Fitting
II
l_∞ error
noise level

Mean curvature and principal directions

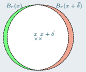


Optimizations with convolution

DGtal

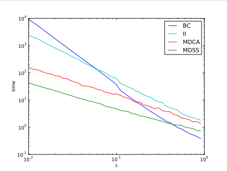
- Open-source C++ library
- Geometry structures, algorithm & tools for digital data
- `http://libdgtal.org`

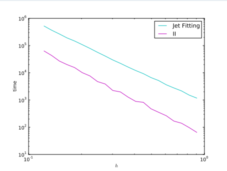
Optimization with displacement masks



Complexity :

- without optimization : $O((r/h)^{d})$
- with optimization : $O((r/h)^{d-1})$





Plan

Conclusion & Future work

- Integral Invariant is perfect for digital geometry
- A unique estimator for both 2D and 3D
- Easy to implement
- Fast computation with masks
- Convergent with a least a uniform convergence speed in $O(h^{\frac{1}{3}})$
- Needs a parameter (r for the kernel radius)

Future work

- Scale-Space analysis
- Feature detection

Choice of radius